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Some Particular Norm in the Sobolev Space $H^1[a, b]$

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Abstract

This paper is a continuation of the recent paper of the author, where a certain reproducing kernel Hilbert space $X_{\mathcal{S}}$ was constructed. The norm in $X_{\mathcal{S}}$ is related to a certain generalized isoperimetric inequality in $\mathbb{R}^2$. In the present paper we give an alternative description of the space $X_{\mathcal{S}}$, which appears to be a Sobolev space $H^1[a, b]$ with some special norm.

Keywords Absolutely continuous functions · Isoperimetric inequality · Hilbert spaces · Sobolev spaces · Reproducing kernels

Mathematics Subject Classification 46E22

1 Introduction

This paper is a continuation of the recent paper of the author, [8]. In this paper we consider a Sobolev space $H^1[a, b]$. We recall some necessary definitions and properties of these spaces below, but at the beginning it is sufficient to know that the elements of $H^1[a, b]$ are absolutely continuous functions with the derivatives in $L_2[a, b]$. Then, if one will define an unitary structure in $H^1[a, b]$, the formula for $\langle f, g\rangle$ must necessarily contain the bilinear form ($f, g \in H^1[a, b]$)

$$\int_a^b f'(x) \cdot g'(x)dx. \tag{1}$$

Clearly the quadratic form generated by (1) is only a seminorm in $H^1[a, b]$, hence to have a norm one must add "something" to (1), or consider a smaller space. In (11)

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we present a typical example of this kind. In this paper we propose, a new formula to define an inner product in $H^1[a, b]$ inspired by the very old idea of the isoperimetric inequality in the plane. The paper [8] contains the construction of a certain reproducing kernel Hilbert space $X_{\mathcal{S}}$ (the necessary information concerning these class of spaces are to be found in the next section). Its kernel, given by the formula (7), is not very complicated. However the description of the function space $X_{\mathcal{S}}$ is not satisfactory. The elements of this space are the equivalent classes of pairs of convex sets relative to some equivalence relation [formula (2)] and then it is not easy to look on the elements of $X_{\mathcal{S}}$ as on the real function defined on some interval. But, on the other hand, we have a concrete positively defined function K [in our situation the function $K(\varphi, \psi)$ given by the formula (7)] so we may try to construct another, perhaps more convenient, model of the space $\mathcal{H}(K)$ corresponding to the considered kernel. Such a situation occurs frequently in the theory of RKHS's and has even his own name (*the reconstruction problem,* cf [4]). The aim of this paper is to give such a concrete description of the space $\mathcal{H}(K)$ for the kernel given by (7). It will appear that in our case the space $\mathcal{H}(K)$ is a Sobolev space $H^1[a, b]$ equipped with some special norm. The construction of "our" $\mathcal{H}(K)$ is contained in the three theorems: Theorem 4, Theorem 5 and the Theorem 6.

In Theorem 4 we define a certain bilinear form $\langle , \rangle_i$ (14) and we prove that this form is positively defined. In the proof we use the Wirtinger inequality (3) in a general version for the derivatives from $L_2[a, b]$. This variant of the Wirtinger inequality is to be found in [3].

In Theorem 5 we check the reproducing property of the given kernel (7). The proof is easy to understand, but may be is heavy to read because of lengthy calculations. The kernel (7), as far as we know, was not studied before.

In the last Theorem 6 we prove, that the constructed space is an RKHS. For this we must check, that the evaluation functionals are commonly bounded, and that the constructed space is complete. This type of arguments is typical in examples of Sobolev spaces $H^1[a, b]$ [2]. Summarizing, the main result of this paper says, that the space $H^1[-\frac{\pi}{2}, \frac{\pi}{2}]$ equipped with the norm $|| \cdot ||_i$ is a reproducing kernel Hilbert space corresponding to the same kernel (7) as the space $X_{\mathcal{S}}$ constructed in [8].

We present also one more construction of the considered space $\mathcal{H}(K)$, called here the *sequence model*. This sequence model, if one looks from strictly theoretical point of view, does not bring new information, but gives tools for proofs and for numerical calculations. This model represents also a kind of presentation of the density of polygons in the space of convex sets.

From a number of known examples of the kernels in the space $H^1[a, b]$, we present here one see formulas (12) and (13) in order to compare it with the considered here kernel (7).

This paper is organized as follows. In the second part we give a summary of the paper [8] necessary to understand the main result of the present paper. We recall also some information on the Sobolev of the type $H^1[a, b]$ in the aim to see the "particularity" of the norm we are going to construct. In the third section we define some special (particular) norm $|| \cdot ||_i$ in the space $H^1[-\frac{\pi}{2}, \frac{\pi}{2}]$. In the last section we present the construction of the *sequence model*.

2 About the Space of Generalized Convex Sets

In this subsection we recall the construction of the space $X_{\mathcal{S}}$ from [8]. Let $\mathcal{S}$ denote the cone of convex, compact and centrally symmetric sets in the plane $\mathbb{R}^2$. In this cone we consider the *Minkowski addition* and the scalar multiplication by non-negative real numbers. In the product $\mathcal{S} \times \mathcal{S}$ we consider the equivalence relation

$$(U, V)\diamondsuit(P, Q) \Longleftrightarrow U + Q = V + P \tag{2}$$

It is known, that the quotient $\mathcal{S} \times \mathcal{S}/\diamondsuit$ is a real vector space (for details see e.g. [5]), and the space $X_{\mathcal{S}}$ mentioned above is the completion of the quotient $\mathcal{S} \times \mathcal{S}/\diamondsuit$ with respect to the so-called *isoperimetric norm*, which is constructed in a few steps. At first we prove, that the two dimensional Lebesgue measure $m : \mathcal{S} \longrightarrow [0, \infty)$ can be extended in a unique way to the polynomial of the second degree $m^* : \mathcal{S}\times\mathcal{S}/\diamondsuit \longrightarrow \mathbb{R}$ and the functional $o : \mathcal{S} \longrightarrow \mathbb{R}$ where $o(U)$ is a perimeter of U, can be extended to a linear functional on the whole $\mathcal{S} \times \mathcal{S}/\diamondsuit$. Next we prove that for this extended measure m^* the *generalized isoperimetric inequality* holds, which means that for each $[U, V] \in \mathcal{S} \times \mathcal{S}/\diamondsuit$ we have

$$(o(U) - o(V))^2 - 4\pi \cdot m([U, V]) \geq 0, \tag{3}$$

where $o(U) - o(V)$ is the perimeter of the pair $[U, V]$ and (here and in the sequel) we write $m([U, V])$ instead of $m^*([U, V])$. Using this fact it was proved, that the formula

$$||[U, V]||_i^2 = 2 \cdot (o(U) - o(V))^2 - 4\pi \cdot m([U, V]) \tag{4}$$

is a unitary norm in $\mathcal{S} \times \mathcal{S}/\diamondsuit$. Finally the space $X_{\mathcal{S}}$ was defined as the completion of the unitary space $(\mathcal{S} \times \mathcal{S}/\diamondsuit; || \cdot ||_i)$.

2.1 About the RKHS

In the last section of [8] we prove that the space $X_{\mathcal{S}}$ is a *reproducing kernel Hilbert space* (RKHS for short). Although the reproducing property was discovered by Zaremba more than hundred years ago (1906), the first systematic lecture is due to Aronszajn in 1950 (see [1]). The necessary information concerning RKHS are to be found in the book of Berlinet and Thomas-Agnant [2] or in the recent book of Szafraniec [6]. The RKHS are function spaces, so at first one must recognize the elements of $\mathcal{S} \times \mathcal{S}/\diamondsuit$ as real functions on the interval $\Delta = [-\frac{\pi}{2}, \frac{\pi}{2}]$. This may be done as follows. We consider the functions

$$I_\psi : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ni \varphi \longrightarrow \sin|\varphi - \psi|, \tag{5}$$

where $\varphi, \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. These functions I_ψ are called *diangles*. Each diangle defines a line in $\mathbb{R}^2$ $(\mathbb{R} \cdot (\cos\psi, \sin\psi))$ which will be denoted also by I_ψ and for the set

$U \in \mathcal{S}$ the number $\overline{U}(I_\psi)$ denotes the *width* of the set U with respect to the line I_ψ. The correspondence

$$[U, V] \to f_{[U,V]} = \overline{U}(I_\varphi) - \overline{V}(I_\varphi), \tag{6}$$

allows us to see the equivalent classes $[U, V]$ as functions on the interval Δ (the details in [8]). Now we are ready to formulate the main result from [8].

Theorem 1 *The space $X_{\mathcal{S}}$ with the norm (4) is a reproducing kernel Hilbert space with the kernel $K : \Delta \times \Delta \longrightarrow \mathbb{R}$, where*

$$K(\varphi, \psi) = 2 - \frac{\pi}{2} \sin |\varphi - \psi|. \tag{7}$$

The elements of $X_{\mathcal{S}}$ are, roughly speaking, the "differences" of convex sets (differences with respect to the Minkowski addition). This makes it possible to understand their geometrical character, but on the other hand, given a function $f : \Delta \longrightarrow \mathbb{R}$ it is difficult to prove (or disprove), that $f \in X_{\mathcal{S}}$. In this paper we will solve this problem. More exactly we will prove, that $X_{\mathcal{S}}$ is a certain Sobolev space.

2.2 About the Sobolev spaces

We shall start by recalling some commonly known definitions and theorems concerning the Sobolev spaces.

Definition 2 Let $\Delta = [a, b] \subset \mathbb{R}$ be a compact interval. A function $f : \Delta \longrightarrow \mathbb{R}$ is said to be *absolutely continuous* on Δ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{p} |f(b_k) - f(a_k)| \leq \epsilon \tag{8}$$

for every finite number of nonoverlappig intervals (a_k, b_k), $k = 1, 2, \ldots, p$ with $[a_k, b_k] \subset \Delta$ and

$$\sum_{k=1}^{p} (b_k - a_k) \leq \delta. \tag{9}$$

Let $AC_1(\Delta)$ denote the vector space of all absolutely continuous functions on Δ. It is known, that each $f \in AC_1(\Delta)$ is differentiable almost everywhere and its derivative f' is a Lebesgue integrable function (i.e. $f' \in L_1(\Delta)$). We will consider a subspace $H^1(\Delta) \subset AC_1(\Delta)$ claiming that

$$f \in H^1(\Delta) \Longleftrightarrow f \in AC_1(\Delta) \wedge f' \in L_2(\Delta). \tag{10}$$

It is also known (see e.g. [4]), that one may consider an inner product $\langle , \rangle$ in $H^1(\Delta)$ given by the formula (11) for $f, g \in H^1(\Delta)$:

$$\langle f, g \rangle = \int_a^b (f \cdot g + f' \cdot g') \tag{11}$$

which is well defined since $f', g' \in L_2(\Delta)$. One may check, that $H^1(\Delta)$, equipped with the above inner product (11), is a Sobolev space with the reproducing kernel $K(x, y)$ where

$$K(x, y) = \frac{\cosh(x-a)\cosh(b-y)}{\sinh(b-a)}, \ a \le x \le y \le b \tag{12}$$

$$K(x, y) = \frac{\cosh(x-a)\cosh(b-y)}{\sinh(b-a)}, \ a \le y \le x \le b \tag{13}$$

Some other examples of reproducing kernels in Sobolev spaces are to be found in [2] or in [4].

In the present paper we construct another inner product $\langle , \rangle_i$ defined in a subspace $H_0^1(\Delta) \subset H^1(\Delta)$, related to another reproducing kernel $K_i(x, y)$, which will be defined in the next section.

3 An Inner Product in the Sobolev Space $H_0^1(\Delta)$

Now we fix $\Delta = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and we consider the subspace $H_0^1 \subset H^1(\Delta)$ claiming that $f(-\frac{\pi}{2}) = f(\frac{\pi}{2})$. This means, that $H_0^1(\Delta)$ may be identified with the space of periodic functions with analogous properties (i.e. absolutely continuous and with the derivatives in $L_2(\Delta)$. For $f, g \in H_0^1(\Delta)$ we set

$$\langle f, g \rangle_i = \frac{1}{\pi^2}\left[2 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right) \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\right) - \pi \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f \cdot g - f' \cdot g')\right]. \tag{14}$$

Here, and below we will often write $\int_a^b f$ or $\int_a^b f(x)$ instead of $\int_a^b f(x)dx$. We will frequently also write $\int f$ instead of $\int_a^b f$ in the case when $a = -\frac{\pi}{2}$ and $b = \frac{\pi}{2}$.

It is clear, that the formula (14) is well defined (because of $f', g' \in L_2(\Delta)$) and defines a bilinear form in $H_0^1(\Delta)$. The coefficient $\frac{1}{\pi^2}$ is clearly without importance and is chosen to have $\langle \mathbf{1}, \mathbf{1} \rangle_i = 1$ (where $\mathbf{1}$ denotes the constant (and equal 1) function). Hence, to have a norm related to the form (14), it remains to show that $\langle , \rangle_i$ is positively defined. The proof of this positivity is similar in fact to the known proof of the isoperimetric inequality based on the Wirtinger inequality.

Let us start by recalling a variant of the Wirtinger inequality (for the proof and some other formulations see [3,7]).

Theorem 3 *Let f be a continuous and periodic function with the period π. Let $f' \in L_2(\Delta)$ and let*

$$\overline{f} = \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right), \tag{15}$$

Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f')^2 \geq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f - \overline{f})^2, \tag{16}$$

and equality holds if and only if $f = a\cos(t) + b\sin(t)$.

Now we have the following

Theorem 4 *Let* $f \in H_0^1(\Delta)$. *Then*

$$\langle f, f\rangle_i = 2\left(\int_{-\pi/2}^{\pi/2} f\right)^2 - \pi\int_{-\pi/2}^{\pi/2} ((f)^2 - (f')^2) \geq 0. \tag{17}$$

and the equality holds only when $f = 0$. *In other words, the form* (14) *is positively defined.*

Proof Let us set, as above

$$\overline{f} = \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right), \tag{18}$$

and let

$$E(f) = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right)^2 - \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ((f)^2 - (f')^2) \tag{19}$$

We want to show, that $E(f) \geq 0$. Consider the difference

$$F(f) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f')^2 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f - \overline{f})^2. \tag{20}$$

The Wirtinger inequality says precisely that $F(f) \geq 0$. This implies that

$$E(f) = (\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f)^2 - \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ((f)^2 - (f')^2) \geq 0. \tag{21}$$

Indeed

$$\begin{aligned} 0 \leq F(f) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f')^2 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f)^2 + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot \left(\frac{1}{\pi} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right) \\ - \pi \cdot \left(\frac{1}{\pi^2} \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right)^2\right) \end{aligned}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f')^2 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f)^2 + \frac{1}{\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \right)^2 ,$$

and this is equivalent to the desired inequality $E(f) \geq 0$. Then $\langle f, f \rangle_i \geq 0$.

If $\langle f, f \rangle_i = 0$ then

$$\langle f, f \rangle_i = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \right)^2 + E(f) = 0$$

and hence $\overline{f} = 0$ and $E(f) = 0$. Thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ((f)^2 - (f')^2) = 0$$

and by (17) $F(f) = 0$, which means that the equality holds in the Wirtinger inequality and additionally $\overline{f} = 0$. We know that in such a case $f = a\cos(t) + b\sin(t)$ and moreover $\overline{f} = 0$. Since $f(-\frac{\pi}{2}) = f(\frac{\pi}{2})$ then $b = 0$ and since $\overline{f} = 0$ then $a = 0$. Hence $f = 0$.

In consequence we have proved that the $H_0^1(\Delta)$ with the inner product (14) is an unitary space. The norm induced by (14) is

$$\begin{aligned} ||f||_i^2 &= \frac{1}{\pi^2} \cdot \left(2 \cdot \left(\int_{-\pi/2}^{\pi/2} f \right)^2 - \pi \int_{-\pi/2}^{\pi/2} ((f)^2 - (f')^2) \right. \\ &= \frac{1}{\pi^2} (o^2(f) + E(f)), \end{aligned} \tag{22}$$

where

$$o^2(f) = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \right)^2 .$$

This ends the proof of Theorem 4. □

We will prove now the reproducing property.

Theorem 5 *The function $K : [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$ defined by*

$$K_i(x, y) = 2 - \frac{\pi}{2} \cdot \sin|x - y|, \tag{23}$$

is a reproducing kernel in the unitary space $H_0^1(\Delta)$. This function will be called the isoperimetric kernel.

Proof Let $k_y(x) = 2 - \frac{\pi}{2} \cdot \sin|x - y|$, (i.e. $k_y = K_i(\cdot, y)$ is a kernel function). We will check (this is the so-called reproduction property) that for each function $f \in H_0^1(\Delta)$ and for each $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have:

$$f(y) = \langle f, k_y \rangle_i \tag{24}$$

Let us fix y and f as above. We must verify that

$$f(y) = \frac{1}{\pi^2}\left(2 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k_y\right) - \pi \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f \cdot k_y - f' \cdot k_y')\right). \tag{25}$$

Let us denote

$$\begin{aligned} A &= 2 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k_y\right), \\ B &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot k_y, \\ C &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f' \cdot k_y'. \end{aligned}$$

We must check that

$$f(y) = \frac{1}{\pi^2}(A - \pi B + \pi C).$$

Let us recall, that in the calculations which will be done below, we will frequently write $\int h$ instead of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(x)dx$.

We have

$$\begin{aligned} A &= 2 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right) \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2 - \frac{\pi}{2}\sin|x-y|\right) = 2 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right) \cdot (2\pi - \pi) \\ &= 2\pi \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\right). \end{aligned} \tag{26}$$

Now we will compute B.

$$\begin{aligned} B &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot \left(2 - \frac{\pi}{2} \cdot \sin|x-y|\right) \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f - \frac{\pi}{2} \cdot \int f \cdot \sin|x-y| = 2\int f - \frac{\pi}{2} \cdot \int f \cdot \sin|x-y| \\ &= 2\int f - \frac{\pi}{2} \cdot \left(\int_{-\frac{\pi}{2}}^{y} f \cdot \sin(y-x) + \int_{y}^{\frac{\pi}{2}} f \cdot \sin(x-y)\right) \end{aligned} \tag{27}$$

The case of C is more complicated. We have

$$C = \int f' \cdot k_y' = \int f' \cdot \left(-\frac{\pi}{2} \cdot \sin|x-y|\right)$$

$$= \int_{-\frac{\pi}{2}}^{y} \left(-\frac{\pi}{2}\right) f' \cdot (-\cos(x-y)) + \int_{y}^{\frac{\pi}{2}} \left(-\frac{\pi}{2}\right) f' \cos(x-y)$$
$$= \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{y} f' \cos(x-y) - \frac{\pi}{2} \int_{y}^{\frac{\pi}{2}} f' \cos(x-y).$$

Now we apply twice the integration by parts and we obtain that the above is equal

$$= \frac{\pi}{2} \left[f \cos(x-y)|_{-\frac{\pi}{2}}^{y} + \int_{-\frac{\pi}{2}}^{y} f \sin(x-y) - f \cos(x-y)|_{y}^{\frac{\pi}{2}} \right.$$
$$\left. - \int_{y}^{\frac{\pi}{2}} f \sin(x-y) \right]$$
$$= \frac{\pi}{2} \left[1 \cdot f(y) - f\left(\frac{-\pi}{2}\right) \cos\left(-\frac{\pi}{2} - y\right) - f\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2} - y\right) + 1 \cdot f(y) \right.$$
$$\left. - \left(\int_{\frac{-\pi}{2}}^{y} f(-\sin(x-y)) + \int_{y}^{\frac{\pi}{2}} f(-\sin(x-y)) \right) \right].$$

Since $f(-\frac{\pi}{2}) = f(\frac{\pi}{2})$ then we obtain

$$\frac{\pi}{2} \left[2f(y) - f\left(\frac{\pi}{2}\right) \left(\cos\left(\frac{\pi}{2} + y\right) + \cos\left(\frac{\pi}{2} - y\right) \right) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot \sin|x-y| \right].$$

But $\cos(\frac{\pi}{2} + y) = -\sin(y)$ and $\cos(\frac{\pi}{2} - y) = \sin(y)$ so finally we have

$$C = \frac{\pi}{2} \left[2f(y) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot \sin|x-y| \right].$$

Hence

$$\begin{aligned}
\langle f, k_y \rangle_i &= \frac{1}{\pi^2} [A - \pi B + \pi C] \\
&= \frac{1}{\pi^2} \left[2\pi \left(\int f \right) - \pi \left(2 \left(\int f \right) - \frac{\pi}{2} f \cdot \sin|x-y| \right) \right. \\
&\quad \left. + \pi \frac{\pi}{2} \left(2f(y) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \cdot \sin|x-y| \right) \right] \\
&= \frac{1}{\pi^2} \left[2\pi \left(\int f \right) - 2\pi \left(\int f \right) \right. \\
&\quad \left. + \frac{\pi^2}{2} \int f \sin|x-y| + \pi^2 f(y) - \frac{\pi^2}{2} \int f \sin|x-y| \right] \\
&= \frac{1}{\pi^2} \cdot \pi^2 f(y) = f(y).
\end{aligned}$$

This ends the proof of the Theorem 5 □

It remains to prove now the main result of this section, which says that $H_0^1(\Delta)$ with the norm (22) is an RKHS.

Theorem 6 *With the notations as above, the space $H_0(\Delta)$ is a reproducing kernel Hilbert space and its kernel is given by*

$$K_i(x, y) = 2 - \frac{\pi}{2} \sin |x - y|. \tag{28}$$

Proof The argumentation is similar for example to that in [4]. First we check that the evaluation functionals $e_x(f) = f(x)$ are bounded with respect to the isoperimetric norm. Since $H_0(\Delta) = H_{00} + \mathbb{R} \cdot \mathbf{1}$ then it is sufficient to prove, that each e_x is bounded on the subspace $H_{00} \subset H_0^1$, composed of those functions which vanish at 0, (i.e. we may additionally assume, that $f(0) = 0$).

Let us suppose that $f \in H_0^1(\Delta)$. Let $x + h, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. By the reproducing property (just proved) we have:

$$f(x + h) - f(x) = \langle f, k_{x+h}\rangle_i - \langle f, k_x\rangle_i = \langle f, k_{x+h} - k_x\rangle_i,$$

or

$$\begin{aligned} k_{x+h}(t) - k_x(t) &= \Big(2 - \frac{\pi}{2} \sin |t - (x + h)|\Big) - \Big(2 - \frac{\pi}{2} \sin |t - x|\Big) \\ &= \frac{\pi}{2}(I_x(t) - I_{x+h}(t)), \end{aligned}$$

where I_ψ is a diangle (see (5), [8]). More exactly I_ψ is a function given by the formula $I_\psi(\varphi) = \sin |\varphi - \psi|$. Hence we may write (using Schwarz inequality)

$$|f(x + h) - f(x)| = |\frac{\pi}{2}\langle f, I_x - I_{x+h}\rangle_i| \leq \frac{\pi}{2}||f||_i \cdot ||I_x - I_{x+h}||_i. \tag{29}$$

Now we may compute the isoperimetric norm of the difference of diangles $I_x - I_{x+h}$. Namely, using the formula for the isoperimetric norm from [8] we have:

$$\begin{aligned} ||I_{x+h} - I_x||_i^2 &= \frac{1}{4\pi^2}\Big[2(o(I_x) - o(I_{x+h}))^2 - 4\pi m([I_{x+h}, I_x])\Big] \\ &= \frac{1}{4\pi^2} \cdot 4\pi \cdot 4 \cdot 1 \cdot 1 \cdot \sin |h| = \frac{4}{\pi} \sin |h|. \end{aligned} \tag{30}$$

However, if we want to have an independent proof of Theorem 6, we must compute the norm $||I_{x+h} - I_x||^2$ using formula (17). A direct calculation of the definite integrals which are in (17) is perhaps not to difficult, but is rather time-absorbing. Thankfully, we have the reproducing property (just proved). Hence

$$\begin{aligned} ||I_\varphi - I_\psi||^2 &= \langle I_\varphi - I_\psi, I_\varphi - I_\psi\rangle_i = \frac{2}{\pi}\langle I_\varphi - I_\varphi, \frac{\pi}{2} I_\psi - 2, 2 - \frac{\pi}{2} I_\psi\rangle_i \\ &= \frac{2}{\pi}\langle I_\varphi - I_\psi, -k_\psi + k\varphi\rangle_i \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left(\langle I_\varphi, -k_\varphi\rangle_i + \langle I_\psi, k_\varphi\rangle_i + \langle I_\varphi, k_\psi\rangle_i - \langle I_\psi, k_\psi\rangle_i\right)\\
&= \frac{2}{\pi}(-0 + I_\varphi(\psi) + I_\psi(\varphi) + 0)\\
&= \frac{4}{\pi}\sin|\varphi - \psi|.
\end{aligned}
$$

Setting $h = \varphi - \psi$ we obtain the desired equality. In consequence

$$||I_{x+h} - I_x||_i \le \sqrt{\frac{4}{\pi}}\sqrt{|h|}, \tag{31}$$

and hence

$$|f(x+h) - f(x)| \le \sqrt{\pi}\cdot\sqrt{|h|}(||f||_i). \tag{32}$$

Setting $x = 0$ and taking f such that $f(0) = 0$ we obtain

$$|f(h)| \le \sqrt{\pi}\cdot\sqrt{|h|}(||f||_i).$$

Hence all evaluation functionals $(e_t)_{t\in\Delta}$ are bounded. Moreover the norm of uniform convergence on Δ is weaker than the convergence with respect to the isoperimetric norm.

To end the proof of Theorem 6 it remains to show, that $H_0^1(\Delta)$ with the isoperimetric norm is complete. Take then a sequence (f_n), which is a Cauchy sequence with respect to the isoperimetric norm. Hence

$$\pi^2||f_n - f_m||^2 = o^2(f_n - f_m) + E(f_n - f_m).$$

By our assumption both sequences $o^2(f_n - f_m)$ and $E(f_n - f_m)$ tend to 0 (when $m, n \to \infty$). But

$$E(f_n - f_m) = o^2(f_n - f_m) - \pi\left(\int(f_n - f_m)^2 - \int(f_n' - f_m')^2\right).$$

Since $o^2(f_n - f_m)$ tends to 0 then $(\int(f_n - f_m)^2 - \int(f_n' - f_m')^2)$ tends to 0. But, by the remark made above, the sequence (f_n) being a Cauchy sequence with respect to the isoperimetric norm is also a Cauchy sequence with respect to the norm of uniform convergence on the compact interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence $\int(f_n - f_m)^2$ tends to 0 and in consequence $\int(f_n' - f_m')^2)$ tends to 0. This means, in particular, that (f_n') is a Cauchy sequence in $L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$, then f_n' tends to a function h from $L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$ with respect to the L^2 norm. But f_n tends uniformly to a continuous function f. Since

$$f(x) = \lim_n f_n(x) = \lim_n \int_{-\frac{\pi}{2}}^{x} f_n'(t)dt = \int_{-\frac{\pi}{2}}^{x} h(t)dt,$$

then f is absolutely continuous and since $f_n(-\frac{\pi}{2}) = f_n(\frac{\pi}{2})$ then also $f(-\frac{\pi}{2}) = f(\frac{\pi}{2})$. Thus $f \in H_0^1(\Delta)$ and $||f_n - f||_i$ tends to 0. Hence $H_0^1(\Delta)$ is complete with respect to the isoperimetric norm. □

We will end this chapter by a number of remarks.

Remark 7 The last theorem asserts, that the space $H_0(\Delta)$ with the norm

$$||f||_i^2 = \frac{1}{\pi^2} \cdot \left(2 \cdot \left(\int_{-\pi/2}^{\pi/2} f\right)^2 - \pi \int_{-\pi/2}^{\pi/2} ((f)^2 - (f')^2)\right)$$
$$= \frac{1}{\pi^2}(o^2(f) + E(f)) \tag{33}$$

is isometric to the space $X_{\mathcal{S}}$ of generalized convex sets with the norm

$$||[U, V]||^2 = \frac{1}{4\pi^2}[2o^2([U, V]) - 4\pi \cdot m([U, V]), \tag{34}$$

where $o([U, V]) = o(U) - o(V)$ is the generalized perimeter and

$$m([U, V]) = 2m(U) + 2m(V) - m(U + V) \tag{35}$$

is the generalized Lebesgue measure in the plane. This is true since both spaces have the same (isoperimetric) kernel. The details concerning the formulas (34) and (35) are described in [8]. The isometry between $X_{\mathcal{S}}$ and $H_0^1(\Delta)$ is established by the correspondence

$$\mathcal{S} \times \mathcal{S} \ni [U, V] \to f_{[U,V]} \in H_o^1(\Delta)$$

where

$$f_{[U,V})(\varphi) = \overline{U}(I_\varphi) - \overline{V}(I_\varphi),$$

and where $\overline{U}(I_\varphi)$ denotes the width of the set U with respect to the straight line generated by the diangle I_φ, (see [8]).

This means, that the term $\int f = \int f_{[U,V]}$ is (because of the isometry) the perimeter of $[U, V]$ (in particular of U) and the term

$$\int_{-\pi/2}^{\pi/2} (f_{[U,V]})^2 - (f'_{[U,V]})^2 = m([U, V])$$

equals to the measure of $[U, V]$. This fact is known for a long time as the so-called Cauchy formula (true not only for centrally symmetric, but for all convex and compact sets). Let us notice that (6) may be considered as an extension of the Cauchy formulas for generalized measure.

Remark 8 Let us come back to the inequality (32) proved above, and saying that for each $f \in H_0^1(\Delta)$, for each $x \in \Delta$ and $h\rangle 0$ the following inequality holds:

$$|f(x+h) - f(x)| \leq \sqrt{\pi} \cdot ||f||_i \sqrt{h}. \tag{36}$$

Setting $x + h = x_2$, $x = x_1$ and $\sqrt{\pi}||f||_i = C(f)$ we obtain the evaluation

$$|f(x_2) - f(x_1)| \leq C(f)\sqrt{|x_2 - x_1|} = C(f)|x_2 - x_1|^{\frac{1}{2}}.$$

This means, that each function $f \in H_0^1(\Delta)$ (each $[U, V] \in X_{\mathcal{S}}$) satisfies the Hölder condition with the exponent $\frac{1}{2}$. Hence the function

$$u(x) = |x|^{\frac{1}{3}} \notin X_{\mathcal{S}}$$

and the diangles are examples, that the exponent $\frac{1}{2}$ cannot be improved. This is the known Hölder condition in Sobolev spaces and we see that it is not too difficult to prove, when we work in the representation $H_0^1(\Delta)$. On the other hand, this is not easy to observe, if we work with generalized convex sets in $X_{\mathcal{S}}$.

Remark 9 The third remark concerns the coefficient 2 in the formula

$$||[U, V]||^2 = \frac{1}{4\pi^2}[2o^2([U, V]) - 4\pi \cdot m([U, V]),$$

or in the formula (17). The generalized isoperimetric inequality

$$o^2([U, V]) - 4\pi m([U, V]) \geq 0,$$

or more exactly the inequality

$$o^2(x) - 4\pi m(x) \geq 0$$

for $x \in X_{\mathcal{S}}$ gives only a seminorm on $X_{\mathcal{S}}$ with one-dimensional kernel $\mathbb{R} \cdot \mathbf{1}$. Thus as it was remarked in the Introduction, to have the ”true” norm one must add ”something” non-vanishing on the line $\mathbb{R} \cdot \mathbf{1}$. In this (and the previous [8] paper, this added term is the seminorm $o^2([U, V])$ (more exactly $o^2(x)$—a square of the linear functional—for $x \in X_{\mathcal{S}}$). It is clear, that one may consider the family of norms of the form

$$||[U, V]||^2 = \frac{1}{4\pi^2}[\theta \cdot o^2([U, V]) - 4\pi \cdot m([U, V]),$$

where $\theta > 1$.

The corresponding reproducing kernel has the following form:

$$K_\theta(\varphi, \psi) = \theta - \frac{\pi}{2} \sin |\varphi - \psi|,$$

where θ is a greater than 1. For $\theta = 2$ we obtain the kernel K_i. Let us observe additionally, that for $\theta = 1$ we obtain an interesting (since canonical) kernel

$$K_1(x, y) = 1 - \frac{\pi}{2} \sin |x - y|,$$

corresponding to the subspace of $X_o \subset X_{\mathcal{S}}$, which is the kernel of the linear functional (perimeter) $x \longrightarrow o(x)$.

4 A Sequence Model

The construction of the RKHS space $H_0^1(\Delta)$ presented above is frequently called *from space to kernel* (see e.g. [6]). In this section we will construct the same function space—i.e. $X_{\mathcal{S}}$—but in a way, which is called *from kernel to space*. The aim for which we repeat this commonly known construction is that it gives a possibility to see a kind of finite dimensional version of the generalized isoperimetric inequality. As usual, in this model the space $X_{\mathcal{S}}$ will appear to be the completion of a certain function space $\mathcal{F}$ with respect to a suitable inner product (suitable norm), corresponding to a given isoperimetric kernel. The idea we will describe below in details, was used in [8] in many places, but in an implicit form.

4.1 Definition of a Certain Sequence Space

Let, as above $\Delta = [-\pi/2, \pi/2]$ and let us consider the family of diangles, i.e. the family of functions $(I_\varphi)_{\varphi \in \Delta}$, where

$$I_\varphi : \Delta \ni \psi \longrightarrow I_\varphi(\psi) = \sin |\varphi - \psi|. \tag{37}$$

Let $\mathcal{F}$ denote the subspace of the vector space $\mathbb{R}^\Delta$, generated by the constant function $\mathbf{1}$ and by the diangles i.e.

$$\mathcal{F} = \left\{ x_o \cdot \mathbf{1} + \sum_{i=1}^{k} x_i \cdot I_{\varphi_i} : x_i \in \mathbb{R}, k = 1, 2, ... \right\}. \tag{38}$$

Let us observe, that the space $\mathcal{F}$ is exactly the linear space generated by the kernel functions $k_\psi = 2 \cdot \mathbf{1} - \frac{\pi}{2} I_\psi$ of the considered kernel $K_i(\varphi, \psi) = 2 \cdot \mathbf{1} - \frac{\pi}{2} \sin |\varphi - \psi|$. Now we will define a certain bilinear form $\langle , \rangle_i$ in $\mathcal{F}$. Take two elements $x, y \in \mathcal{F}$ where

$$x = x_o \cdot \mathbf{1} + \sum_{i=1}^{k} x_i \cdot I_{\varphi_i},$$

and

$$y = y_o \cdot \mathbf{1} + \sum_{j=1}^{m} y_j \cdot I_{\psi_j}.$$

Since $\langle , \rangle_i$ is claimed to be bilinear, then

$$\langle x, y\rangle_i = \langle x_o \cdot \mathbf{1} + \sum_{i=1}^{k} x_i \cdot I_{\varphi_i}, y_o \cdot \mathbf{1} + \sum_{j=1}^{m} y_j \cdot I_{\psi_j}\rangle_i = x_o \cdot y_o \langle \mathbf{1}, \mathbf{1}\rangle_i$$
$$+ \sum_{j=1}^{m} x_o y_j \langle \mathbf{1}, I_{\psi_j}\rangle_i + \sum_{i=1}^{k} \langle \mathbf{1}, I_{\varphi_i}\rangle_i + \sum_{i=1}^{k}\sum_{j=1}^{m} x_i y_j \langle I_{\varphi_i}, I_{\psi_j}\rangle_i .$$

Setting $\langle \mathbf{1}, \mathbf{1}\rangle_i = k_{oo}$, $\langle \mathbf{1}, I_\varphi\rangle_i = k_{o,\varphi}$, $\langle I_\varphi, I_\psi\rangle_i = k_{\varphi,\psi}$ where $k_{oo}, k_{o,\varphi}, k_{\varphi,\psi}$ are arbitrarily chosen real numbers, one always obtains a bilinear form, but not necessarily positively defined. Let us set

$$k_{oo} = \langle \mathbf{1}, \mathbf{1}\rangle_i = 1, \tag{39}$$
$$k_{o,\varphi} = \langle \mathbf{1}, I_\varphi\rangle_i = \frac{2}{\pi}, \tag{40}$$

and

$$k_{\varphi,\psi} = \langle I_\varphi, I_\psi\rangle_i = \frac{4}{\pi^2}\left(2 - \frac{\pi}{2} \cdot \sin|\varphi_i - \psi_j|\right). \tag{41}$$

Now we set

Definition 10 For the vectors $x, y \in \mathcal{F}$ defined as above we set

$$\langle x, y\rangle_i = x_o y_o + \frac{2}{\pi}\sum_{i=1}^{k} x_i y_o + \frac{2}{\pi}\sum_{j=1}^{m} x_o y_j$$
$$+\frac{4}{\pi^2}\sum_{i=1}^{k}\sum_{j=1}^{m}\left(2 - \frac{\pi}{2} \cdot \sin|\varphi_i - \psi_j|\right) x_i y_j . \tag{42}$$

The coefficients in the formula (42) are chosen in such a way, that one may easily check directly the reproducing property of the form $\langle , \rangle_i$. In other words it is easy to verify, that $\mathcal{F}$ equipped with the scalar product given by the (42) is identical (isometric) with the subspace of the space $X_{\mathcal{S}}$ spanned by the unit disc and all diangles. In consequence $\langle , \rangle_i$ given by (42) is really an inner product (is positively defined) and the completion of $\mathcal{F}$ equals $X_{\mathcal{S}}$.

Setting in (42) $x = y$ we obtain the following formula for the (isoperimetric) norm in $\mathcal{F}$. Namely

$$||x||_i^2 = x_o^2 + \frac{4}{\pi}\sum_{i=0}^{k} x_o x_i + \frac{4}{\pi^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\left(2 - \frac{\pi}{2}\sin|\varphi_i - \varphi_j|\right) x_i x_j . \tag{43}$$

This formula may be rewritten in the form

$$||x||_i^2 = \frac{4}{\pi^2} \cdot \left[\left(\frac{\pi}{2} x_o + \sum_{i=1}^{k} x_i \right)^2 - \left(\sum_{i=1}^{k} x_i \right)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k} \left(2 - \frac{\pi}{2} \sin |\varphi_i - \varphi_j| \right) x_i x_j \right], \tag{44}$$

or in the form

$$||x_|||_i^2 = \frac{4}{\pi^2} \cdot \left[\left(\frac{\pi}{2} x_o + \sum_{i=1}^{k} x_i \right)^2 - \left(\sum_{i=1}^{k} x_i \right)^2 + 2 \sum_{i,j=1}^{k} x_i x_j - \frac{\pi}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sin |\varphi_i - \varphi_j| \Bigg) x_i x_j \right]. \tag{45}$$

Now, since $||x||^2 \geq 0$ then we obtain the following inequality:

$$\left(\frac{\pi}{2} x_o + \sum_{i=1}^{k} x_i \right)^2 + 2 \sum_{i,j=1}^{k} x_i x_j \geq \left(\sum_{i=1}^{k} x_i \right)^2 + \frac{\pi}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sin |\varphi_i - \varphi_j| x_i x_j. \tag{46}$$

This inequality may be considered as *the isoperimetric inequality for sequences*. If we put $x_0 = 0$ we obtain the form

$$2 \sum_{i,j=1}^{k} x_i x_j \geq \frac{\pi}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} (\sin |\varphi_i - \varphi_j|) x_i x_j, \tag{47}$$

or, finally, the form

$$\sum_{i=1}^{k} \sum_{j=1}^{k} (\sin |\varphi_i - \varphi_j|) x_i x_j \leq \frac{4}{\pi} \left(\sum_{i,j=1}^{k} x_i x_j \right), (= \frac{4}{\pi} (\sum_{i=1}^{k} x_i)^2). \tag{48}$$

Let us remark, that in our interpretation, the sum of diangles $W = \sum_{i=1}^{k} x_i I_{\varphi_i}$ represents (for non-negatives x_i) a polygon with sides I_{φ_i}. Since the perimeter of a diangle is $o(I_\varphi) = 4x_i$ then the number $(\sum_{i=1}^{k} x_i)^2$ equals $\frac{1}{16} o^2(W)$, so, for polygon W the last inequality gives

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \sin|\varphi_i - \varphi_j| x_i x_j \leq \frac{1}{4\pi} o^2(W). \tag{49}$$

It is also not hard to check, that the area of the polygon $W = \sum_{i=1}^{k} x_i I_{\varphi_i}$ is given by

$$area(W) = m(W) = \sum_{i=1}^{k}\sum_{j=1}^{k} \sin|\varphi_i - \varphi_j| x_i x_j. \tag{50}$$

Hence the classical isoperimetric inequality for sequences says simply, that the isoperimetric inequality is valid for polygons (with positive sides). Explicitely (for W as above)

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \sin|\varphi_i - \varphi_j| x_i x_j = m(W) \leq \frac{1}{4\pi} o^2(W) = \frac{4}{\pi}\left(\sum_{i=1}^{k} x_i\right)^2. \tag{51}$$

Let us remark, that as we have proved, the last inequality is valid for all and not only for non-negative sequences (x_i).

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